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Nonpositive curvature on the area-preserving diffeomorphism group

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Abstract

A steady ideal fluid flow on a surface corresponds to a geodesic in the area-preserving diffeomorphism group. The sign of the curvature operator along this geodesic has been of interest since Arnold noticed its connection to Lagrangian stability of the flow: nonpositive curvature implies by the Rauch comparison theorem that Lagrangian perturbations grow at least linearly in time.

We obtain a new necessary and sufficient criterion for a steady flow with analytic stream function and isolated zeroes to have nonpositive curvature operator: either the surface is a flat torus, and the fluid flow has constant pressure; or the surface is a sphere, disc, or annulus with a globally-defined polar coordinate system such that the metric is $\mathrm{d}s^2 = \mathrm{d}r^2 + \varphi^2(r)\mathrm{d}\theta^2$. In the latter case, the velocity field must be of the form $X = u(r)\partial_\theta$. Furthermore, the function $Q = (u\varphi')'/u'$ must be defined for every r and satisfy the differential inequality $\varphi Q' + Q^2 \le 1$.

This criterion is proved by using a new formula for the curvature of the area-preserving diffeomorphism group in the rotationally symmetric case, involving only first integrals in one variable, rather than infinite sums or the solution of a PDE.

Elementary consequences of the criterion are also discussed: for example, there are no flows with nonpositive curvature operator on the standard round sphere; and on a flat surface, every rotationally symmetric flow has nonpositive curvature operator. Finally we show that if a steady flow satisfies both this nonpositive curvature criterion and the well-known Eulerian stability criterion of Arnold,

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then all Lagrangian perturbations grow polynomially in time, in the L^2 norm. Thus this is the first time methods of Riemannian geometry have given rigorous information on stability.

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1. Introduction

The discovery by Arnold [2] that the motion of an ideal fluid on a manifold M is given by geodesics on the volume-preserving diffeomorphism group $\mathcal{D}_\mu(M)$ has led to an interest in the curvature of this infinite-dimensional Riemannian manifold. Since Jacobi fields along geodesics represent linearized Lagrangian perturbations of ideal fluid motion, the sign of the curvature gives information about linear stability.

Arnold [2] was the first to compute a formula for the curvature on $\mathcal{D}_\mu(\mathbb{T}^2)$, using Fourier series. Lukatsky [5] used a similar technique to find a formula for the curvature of $\mathcal{D}_\mu(M)$ for any compact surface. Simpler formulas were obtained for $\mathcal{D}_\mu(S^2)$ by Arakelyan-Savvidy [1], Dowker-Wei [4], and Yoshida [12], using special properties of spherical geometry. Other general formulas have been derived using submanifold geometry by Misiołek [7] and using the Jacobi equation by Rouchon [10]. These formulas have generally suffered the drawback of being computationally unwieldy, requiring infinite sums or the solution of partial differential equations, and thus many properties of curvature have been obscured.

One is mainly interested in the sign of the curvature operator along a particular geodesic. If $\tilde{\mathbf{R}}$ denotes the curvature tensor on $\mathcal{D}_\mu(M)$ and X is the velocity field tangent to the geodesic, then the curvature operator $\tilde{\mathbf{R}}_X := Y \mapsto \tilde{\mathbf{R}}(Y, X)X$ appears in the Jacobi equation. We hope to find conditions on X such that $\tilde{\mathbf{R}}_X$ is either nonpositive or nonnegative in all directions.

The case in which $\tilde{\mathbf{R}}_X$ is nonpositive is especially interesting from the view of Lagrangian stability, since for such flows we know by the Rauch comparison theorem that Jacobi fields must grow in time. Therefore, finding criteria for a flow to have such curvature is the only known rigorous way to prove Lagrangian instability using geometric techniques.

Flows generating nonnegative curvature operators are completely understood. Misiołek [7] demonstrated that if X is a Killing field on an arbitrary manifold M, then the curvature operator is nonnegative. Rouchon [10] proved the converse, at least for the special case of a domain in $\mathbb{R}^3$ (the technique is very easily generalized to an arbitrary manifold of any dimension, as shown in the author's dissertation [8]).

For nonpositive curvature, progress has been slower. Arnold [2] showed that for $k \in \mathbb{N}$, the vector field $X = \sin(kx)\partial_y$ on the torus $\mathbb{T}^2$ had nonpositive curvature operator. Misiołek [7] and Lukatsky [6] separately proved the more general result that if X is a divergence-free field on a manifold M with nonpositive curvature and satisfies $\nabla_X X = 0$ (that is, the flow of X consists of geodesics on M), then $\tilde{\mathbf{R}}_X$ is nonpositive. Misiołek called such vector fields "pressure-constant." It was unknown whether there were any other vector fields which would make $\tilde{\mathbf{R}}_X$ nonpositive.

In this paper we demonstrate that there are many choices of X which are not pressure-constant but which have nonpositive curvature operator. We obtain a necessary and sufficient criterion for a vector field X on an orientable surface M to have nonpositive curvature operator, under the condition that X generate a steady fluid flow (i.e. that the corresponding geodesic in $\mathcal{D}_\mu(M)$ is also a 1-parameter subgroup). We also assume for convenience that X has only isolated zeroes and has a real-analytic stream function.

The criterion is that X must be of the form

$$X = u(r)\partial_\theta \tag{1.1}$$

on a rotationally symmetric manifold M with Riemannian metric of the form

$$\mathrm{d}s^2 = \mathrm{d}r^2 + \varphi^2(r)\,\mathrm{d}\theta^2 \tag{1.2}$$

for some functions u and φ. M must be either a torus with a flat metric (so that φ is constant), or a disc, sphere, or annulus. In these latter cases, the quantity

$$Q(r) \equiv \frac{(\mathrm{d}/\mathrm{d}r)(u(r)\varphi'(r))}{u'(r)} \tag{1.3}$$

must be defined for all r and satisfy the differential inequality

$$\varphi(r)Q'(r) + Q^2(r) \leq 1 \quad \text{for all } r. \tag{1.4}$$

The proof has two parts. First, in Section 3 we show that if X generates a steady fluid flow and Y is another divergence-free vector field with $[X, Y] = 0$, then $\langle\langle \tilde{\mathbf{R}}(Y, X)X, Y\rangle\rangle \geq 0$. Thus it is fairly easy to find examples of sections where the curvature is either zero or positive. If $X = \operatorname{sgrad} f$, then $Y = f \operatorname{sgrad} f$ is one example, and we can easily show that the curvature is in fact positive unless $X\langle X, X\rangle \equiv 0$. If $X\langle X, X\rangle \equiv 0$, then the speed of each particle remains constant in time, and we demonstrate that this condition, combined with incompressibility, imply rotational symmetry not only for the flow but for the surface as well. Thus the condition that $\tilde{\mathbf{R}}_X$ be nonpositive implies Eqs. (1.1) and (1.2).

In addition, we can perform a local analysis of this condition near an isolated zero of X, and conclude that any isolated zeroes of X must be elliptic. This implies that the surface is either a sphere, a disc, an annulus, or a torus. In this way we obtain a structure theorem: any flow for which the curvature operator is nonpositive must have a very special rotationally symmetric form. By narrowing down the possibilities this way, we are able to set up the more explicit analysis of the next section.

Surprisingly, the case of the torus is quite different from the other surfaces. The reason is that on the other surfaces, every rotationally symmetric flow is actually a steady solution of the Euler equations. On the torus this is not necessarily the case, due to the nontrivial homology (basically, $\nabla_X X$ is a vector field whose curl vanishes, but on the torus it need not be the gradient of an actual function). Because of this, we can find many other examples of fields Y commuting with X which also yield strictly positive sectional curvature, unless M is actually flat and X is therefore pressure-constant. So this reduces to Misiołek's result, and this is how we get all the previously known examples on the torus. For the other surfaces, we need to analyze the formulas more carefully.

In Section 4 we provide a very explicit new formula for the curvature along a rotationally symmetric flow in terms of first integrals of various explicit functions, which are slightly different in each of the three remaining cases (sphere, disc, and annulus). This is the first time such a formula for the curvature has been obtained, even for a special case such as this, which does not involve either infinite sums or the (implicit) inversion of the Laplacian. The reason we can obtain such a formula is basically because in the rotationally symmetric two-dimensional case, the Laplacian can be inverted quite explicitly.

In Section 5, we show how this formula for the curvature can be written as a sum of nonpositive terms and a term involving the function Q. If Q is defined everywhere and satisfies (1.4), we show that then $\langle\langle \tilde{\mathbf{R}}_X(Y), Y\rangle\rangle \leq 0$ for all Y. If not, we show how to construct a Y which yields positive curvature.

In Section 6, we derive some interesting consequences of the condition (1.4). For example, if $\varphi(r) = r$ (so that M is the standard flat disc or annulus in $\mathbb{R}^2$), then $Q(r) \equiv 1$ and the condition is automatically satisfied; thus for every rotational flow on the flat disc or annulus, $\tilde{\mathbf{R}}_X$ is nonpositive. The case $\varphi(r) \equiv 1$ is the pressure-constant case on an annulus, which yields $Q(r) \equiv 0$, reproducing Misiołek's result. The flat disc, the flat annulus, and the flat cylinder are the only spaces on which every steady rotational flow has nonpositive curvature operator. We show that on every nonflat rotationally symmetric surface, there are some flows which do satisfy the criterion and others which do not.

If the curvature of M is either always positive or always negative, we can determine qualitative criteria for the existence of flows X, with a stream function and isolated zeroes, with $\tilde{\mathbf{R}}_X$ nonpositive. For example, for such a flow, u cannot have a maximum or minimum except when φ vanishes (i.e., where the metric becomes singular, either at the center of a disc or the two poles of a sphere).

In addition, we show that there are no such flows on a positive-curvature sphere. Thus, in a sense, if the curvature of the underlying manifold is sufficiently positive, then the curvature of the volume-preserving diffeomorphism group must also be somewhere positive, at least in any section containing a steady flow.

Finally we discuss consequences for Lagrangian stability. The Rauch comparison theorem, as proven for $\mathcal{D}_\mu(M)$ by Misiołek [7], shows that if the curvature operator $\tilde{\mathbf{R}}_X$ is nonpositive, then all Jacobi fields Y grow at least linearly in time, in the L^2 norm. So we have at least "slow" (polynomial) instability in the Lagrangian sense, uniformly for all Lagrangian perturbations, for flows satisfying this condition, and this was not previously known.

Even though the curvature operator of such a flow is zero in some directions and negative-definite in many, we cannot say that Jacobi fields necessarily grow exponentially in time. This is because the explicit example of the author [9], computing the growth rate of Jacobi fields along plane-parallel Couette flow, shows that linear growth of Jacobi fields is more typical. It had been previously conjectured by many authors that negative-curvature directions would imply "fast" (exponential) Lagrangian instability, but the author's previous work shows that the difference between fast and slow instability cannot be determined by curvature alone, but only by the Eulerian stability of the flow.

One interesting result is that if $X = u(r)\partial_\theta$ satisfies the inequality (5.3) and also the Arnold Eulerian stability criterion

$$\frac{(\mathrm{d}/\mathrm{d}r)((1/\varphi)(\mathrm{d}/\mathrm{d}r)(\varphi^2 u))}{\varphi u} \neq 0,$$

then by a theorem in the author's previous paper [9] we get both a lower bound and an upper bound for the long-time growth of every Jacobi field in L^2: at least $\mathrm{O}(t)$ and at most $\mathrm{O}(t^2)$. So we are guaranteed slow instability in this case.

We conclude in Section 7 with some natural questions inspired by this research, including the generalizations to three dimensions and to nonsteady flows. For an excellent general overview of the subject, see Chapter 4 of Arnold-Khesin [3].

2. Review of geometry formulas

Let M be an orientable surface, possibly with boundary ∂M. The group under composition of diffeomorphisms of M is denoted $\mathcal{D}(M)$. For simplicity we will assume all objects are C^∞.

At a diffeomorphism $\eta \in \mathcal{D}(M)$, the tangent space $T_\eta\mathcal{D}(M)$ consists of elements $U \circ \eta$, where U is a vector field on M. If $\langle \cdot, \cdot \rangle$ is the Riemannian metric on M and μ is the corresponding area 2-form, the Riemannian metric $\langle\langle \cdot, \cdot \rangle\rangle$ on $T_\eta\mathcal{D}(M)$ is given by the formula

$$\langle\langle U \circ \eta, V \circ \eta \rangle\rangle = \int_M \langle U, V \rangle \circ \eta\, \mu, \quad \text{for any vector fields } U \text{ and } V. \tag{2.1}$$

Given a vector field X on M, we may construct a right-invariant vector field $\mathbf{X}$ on $\mathcal{D}(M)$ by defining $\mathbf{X}_\eta = X \circ \eta$ for each $\eta \in \mathcal{D}(M)$. The covariant derivative $\bar{\nabla}$ on $\mathcal{D}(M)$ then satisfies

$$(\bar{\nabla}_{\mathbf{X}}\mathbf{Y})_\eta = (\nabla_X Y) \circ \eta \tag{2.2}$$

on right-invariant vector fields. See Misiołek [7] for details.

Now consider $\mathcal{D}_\mu(M)$, the submanifold of $\mathcal{D}(M)$ consisting of diffeomorphisms η satisfying $\eta^*\mu = \mu$. At any η, the elements of the tangent space $T_\eta\mathcal{D}_\mu(M)$ are of the form $X \circ \eta$, where X is divergence-free and tangent to the boundary. The L^2 metric (2.1) on $\mathcal{D}(M)$ induces a metric on $\mathcal{D}_\mu(M)$ defined by

$$\langle\langle U \circ \eta, V \circ \eta \rangle\rangle \equiv \int_M \langle U, V \rangle \circ \eta\, \mu = \int_M \langle U, V \rangle \mu.$$

This induced metric is right-invariant.

An arbitrary vector field (not necessarily tangent to ∂M) can be orthogonally projected onto the space of divergence-free vector fields tangent to the boundary using the Hodge decomposition. We notice first that the space of gradients of functions on M is the orthogonal complement of $T_{\mathrm{id}}\mathcal{D}_\mu(M)$ in $T_{\mathrm{id}}\mathcal{D}(M)$, since for any $\phi : M \to \mathbb{R}$ and any $V \in T_{\mathrm{id}}\mathcal{D}_\mu(M)$, we have

$$\int_M \langle V, \nabla\phi\rangle\mu = \int_M \operatorname{div}(\phi V)\mu - \int_M \phi \operatorname{div} V\,\mu = \int_{\partial M} \phi\langle V, n\rangle\iota_n\mu = 0.$$

Thus, given a vector field Z, we solve the Neumann boundary value problem

$$\Delta f = \operatorname{div} Z, \qquad \langle \nabla f, n\rangle|_{\partial M} = \langle Z, n\rangle|_{\partial M}$$

to obtain a function f, unique up to a constant, and then define the orthogonal projection $\mathbf{P}(Z)$ as

$$\mathbf{P}(Z) = Z - \nabla f. \tag{2.3}$$

By construction, $\mathbf{P}(Z)$ is divergence-free and tangent to the boundary.

The covariant derivative $\tilde{\nabla}$ on the submanifold $\mathcal{D}_\mu(M)$ is the projection of the covariant derivative $\bar{\nabla}$. For right-invariant fields $\mathbf{X}$ and $\mathbf{Y}$, then, we have

$$(\tilde{\nabla}_{\mathbf{X}}\mathbf{Y})_\eta = \mathbf{P}(\nabla_X Y) \circ \eta. \tag{2.4}$$

The curvature on the volume-preserving diffeomorphism group is denoted $\tilde{\mathbf{R}}$. Since the metric on $\mathcal{D}_\mu(M)$ is right-invariant, so is the curvature, and it is sufficient to perform all computations at the identity. By formula (2.4), the Riemann curvature operator is given by

$$\tilde{\mathbf{R}}_X(Y) \equiv \tilde{\mathbf{R}}(Y, X)X = \mathbf{P}(\nabla_Y \mathbf{P}(\nabla_X X) - \nabla_X \mathbf{P}(\nabla_Y X) + \nabla_{[X,Y]}X). \tag{2.5}$$

The sectional curvature $\tilde{\mathbf{K}}$ of the 2-plane spanned by vectors X and Y in $T_{\mathrm{id}}\mathcal{D}_\mu(M)$ is given by

$$\tilde{\mathbf{K}}(X, Y) = \frac{\langle\langle \tilde{\mathbf{R}}(Y, X)X, Y\rangle\rangle}{\langle\langle X, X\rangle\rangle\langle\langle Y, Y\rangle\rangle - \langle\langle X, Y\rangle\rangle^2}.$$

However we are concerned only with the sign of the sectional curvature, and so the normalizing factor in the denominator is unimportant. Thus we will work with the non-normalized curvature, which we denote by

$$\mathbf{K}(X, Y) = \langle\langle \tilde{\mathbf{R}}(Y, X)X, Y\rangle\rangle,$$

or simply $\mathbf{K}$ if the 2-plane is fixed.

The Euler equation, satisfied by the tangent vector to a geodesic (right-translated to the identity) is

$$\frac{\partial X}{\partial t} + \mathbf{P}(\nabla_X X) = 0.$$

In case X is independent of time, we have the steady Euler equation $\mathbf{P}(\nabla_X X) = 0$, which is often written in the form $\nabla_X X = -\nabla p$, where p is the pressure. In this case, the geodesic is a 1-parameter subgroup of the volume-preserving diffeomorphism group, and the first term of the curvature formula (2.5) vanishes. This simplification is our primary reason for working only with steady solutions of the Euler equation, although typically in studies of stability these are the only ones considered anyway.

3. The nonpositivity structure theorem

In what follows we use the fact that if $\mathbf{P}(\nabla_X X) = 0$ and $[X, Y] = 0$, then most of the terms in the curvature formula (2.5) vanish. We impose the requirement that the steady vector field X have a globally defined stream function f, so that $X = \operatorname{sgrad} f$; this makes it easy to find a commuting vector field Y.

Proposition 3.1. *Suppose M is a two-dimensional manifold, possibly with boundary. Let f be a function on M which is constant on each boundary component of M. Define* $X = \operatorname{sgrad} f$, *and suppose that* $\nabla_X X = -\nabla p$ *for some function p.*

If $\langle\langle \tilde{\mathbf{R}}(Y, X)X, Y\rangle\rangle \le 0$ *for every divergence-free Y tangent to* ∂M, *then* $X\langle X, X\rangle = 0$.

Proof. Define $Y = fX$. Since $X(f) = \langle \operatorname{sgrad} f, \nabla f\rangle = 0$, we know that Y is divergence-free and tangent to ∂M, and also that $[X, Y] = 0$. Then since $\mathbf{P}(\nabla_X X) = 0$ by assumption, formula (2.5) implies

$$\langle\langle \tilde{\mathbf{R}}(Y, X)X, Y\rangle\rangle = -\int_M \langle Y, \nabla_X \mathbf{P}(\nabla_Y X)\rangle \mu = \int_M \langle \mathbf{P}(\nabla_Y X), \mathbf{P}(\nabla_Y X)\rangle \mu.$$

Thus we must have $\mathbf{P}(\nabla_Y X) = 0$, so that $\nabla_Y X = \nabla q$ for some function q. Thus

$$\nabla q = f\nabla_X X = -f\nabla p = -\nabla(fp) + p\nabla f,$$

and therefore $\operatorname{sgrad}(q + fp) = pX$. As a result,

$$0 = \operatorname{div}(pX) = p \operatorname{div} X + X(p) = X(p).$$

So $X(p) = 0$.

Since $\nabla_X X = -\nabla p$, we know $X\langle X, X\rangle = -2X(p) = 0$, and we are done. □

Lemma 3.2. *Suppose X is a divergence-free vector field on a surface satisfying* $X\langle X, X\rangle = 0$ *everywhere. Then any isolated, nondegenerate zero of X must be elliptic* (*i.e. with index* $+1$).

Proof. Choose normal coordinates in a neighborhood of an isolated, nondegenerate zero of X, so that the metric looks like

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{O}(x^2 + y^2),$$

and $X(0, 0) = 0$. Write

$$X = (ax + by)\partial_x + (cx + dy)\partial_y + \mathrm{O}(x^2 + y^2).$$

Then the divergence-free condition implies $d=-a$, while the nondegeneracy condition implies $a^2+bc\neq 0$, and we can compute

$$X\langle X, X\rangle = 2(a^2+bc)[ax^2+(b+c)xy-ay^2]+\mathrm{O}((x^2+y^2)^{3/2}).$$

So we must have $a=0$ and $b=-c$, with $c\neq 0$, which clearly is an elliptic zero. □

In the following we give a structure theorem which severely restricts the types of steady flows that have nonpositive curvature operators. The assumptions that the zeroes of X are isolated and that X has a global stream function are somewhat restrictive, but still allow many flows. It seems likely that more sophisticated techniques could be used to eliminate these assumptions.

Theorem 3.3. *Suppose X is a vector field on an orientable surface M of the form $X=$ sgrad f, with f a function on M having only isolated, nondegenerate zeroes, and constant on each component of ∂M. Suppose that $X\langle X, X\rangle = 0$ everywhere on M.*

Then the following hold:

- *M is either a sphere, a torus, a disc, or an annulus.*
- *M has a globally defined metric of the form $\mathrm{d}s^2=\mathrm{d}r^2+\varphi^2(r)\,\mathrm{d}\theta^2$, where $\theta\in S^1$, and there is some $R>0$ such that*: –
  - *If M is a torus, then $r\in S^1(R)$, the circle with circumference R, and $\varphi(r)$ is periodic in r and nowhere vanishing.*
  - *If M is a sphere, then $r\in[0, R]$ and $\varphi(r)$ vanishes iff $r=0$ or $r=R$.*
  - *If M is a disc, then $r\in[0, R]$ and $\varphi(r)$ vanishes iff $r=0$.*
  - *If M is an annulus, then $r\in[0, R]$ and $\varphi(r)$ is nowhere vanishing.*
- *$X=u(r)\partial_\theta$, with $u(r)$ nowhere vanishing.*

Proof. Since X is tangent to the boundary, the condition that $X\langle X, X\rangle = 0$ implies that X either vanishes everywhere on the boundary or vanishes nowhere. By assumption, the zeroes of X are isolated, and thus X cannot vanish on the boundary. So we can use the standard Hopf–Poincaré theorem, which says that the sum of the indices of X is the Euler characteristic of the manifold. Since the indices are all $+1$ by Lemma 3.2, the Euler characteristic can only be 2, 1, or 0. Thus if X has two zeroes, then M must be a sphere. If X has one zero, M must be a disc. If X has no zeroes, then M is either an annulus or a torus.

Let E be the unique unit vector field which is everywhere perpendicular to X, with $\mu(E, X)>0$. (E is defined at every point except at the two possible zeroes of X.) Then the divergence of X is given by

$$\operatorname{div} X = \langle\nabla_E X, E\rangle + \frac{1}{\langle X, X\rangle}\langle\nabla_X X, X\rangle = 0,$$

and thus since $\langle\nabla_X X, X\rangle = 0$, we must have $\langle\nabla_E X, E\rangle = 0$. The consequence is that $\langle\nabla_E E, X\rangle = 0$.

Since we obviously have $\langle\nabla_E E, E\rangle = 0$, we therefore know that $\nabla_E E = 0$. This implies that the integral curves of E are geodesics. We will define the radial coordinate r to be the parameter along each geodesic, so that $E = \partial/\partial r$. If M is a sphere or a disc, r will be zero at one of the zeroes of X. On an annulus, we set $r = 0$ on one boundary component. On a torus, the zero set of r can be any integral curve of X.

Since $X = \operatorname{sgrad} f$, we know that every nonsingular integral curve is a level set of f, and thus is diffeomorphic to a circle. The equation $\langle X, \partial/\partial r\rangle = 0$ implies that each curve is a level set of r as well. For each r_0, the flow of X maps any point on the circle $r = r_0$ to itself after a time $T(r) > 0$. Fix a radial geodesic, and define the angular coordinate θ to be the flow of each point on this geodesic for time θ under the vector field $(T(r)/2\pi)X$. Then we have, for each r_0, a diffeomorphism of the standard circle $S^1(2\pi)$ to the level set $r = r_0$.

In these coordinates the metric takes the form

$$\mathrm{d}s^2 = \mathrm{d}r^2 + \varphi^2(r, \theta)\,\mathrm{d}\theta^2.$$

Defining $u(r) = 2\pi/T(r)$, we find that $X = u(r)\partial_\theta$. Thus the condition $X\langle X, X\rangle = 0$ implies that φ is a function of r alone.

All that remains is to check the stated behavior of φ and u at an isolated zero of X. In a neighborhood of an isolated zero, our coordinate system coincides with Riemannian normal coordinates, and by the usual smoothness requirements, we can see that φ must vanish to first order at a zero of X. Thus u cannot also vanish at a zero of X, by nondegeneracy. □

It is easy to see that the converse of this theorem is true as well: if X and M have the properties stated in the conclusion, then X will be a steady Euler flow. The only case in which this might not work is if M is a torus; in that case, $\nabla_X X = -u^2(r)\varphi(r)\varphi'(r)\partial_r$, and this is only the gradient of a function on the torus if the condition

$$\int_0^R u^2(r)\varphi(r)\varphi'(r)\,\mathrm{d}r = 0$$

holds, since the pressure function must also have period R.

This odd property of the nonflat torus actually enables us to eliminate it as a candidate for having a flow with nonpositive curvature operator.

Proposition 3.4. *If $X = u(r)\partial_\theta$ is a steady solution of the Euler equation on a torus $\mathbb{T}^2 = S^1(R) \times S^1(2\pi)$ with metric $\mathrm{d}s^2 = \mathrm{d}r^2 + \varphi^2(r)\,\mathrm{d}\theta^2$, then the curvature operator $\tilde{\mathbf{R}}_X$ is nonpositive if and only if φ is constant.*

Proof. Let $Y = v(r)\partial_\theta$ for some function $v(r)$. Then $[X, Y] = 0$, so that the curvature of the diffeomorphism group reduces to

$$\langle\langle \tilde{\mathbf{R}}(Y, X)X, Y\rangle\rangle = \int_M \langle \mathbf{P}(\nabla_Y X), \mathbf{P}(\nabla_Y X)\rangle\mu.$$

We compute that

$$\begin{aligned}\nabla_Y X &= -v(r)u(r)\varphi(r)\varphi'(r)\frac{\partial}{\partial r}\\ &= \nabla\left(-\int_0^r vu\varphi\varphi'\,\mathrm{d}\rho + \frac{r}{R}\int_0^R vu\varphi\varphi'\,\mathrm{d}\rho\right) - \left(\frac{1}{R}\int_0^R vu\varphi\varphi'\,\mathrm{d}\rho\right)\frac{\partial}{\partial r},\end{aligned}$$

so that

$$\mathbf{P}(\nabla_Y X) = -\left(\frac{1}{R}\int_0^R v(\rho)u(\rho)\varphi(\rho)\varphi'(\rho)\,\mathrm{d}\rho\right)\frac{\partial}{\partial r}.$$

An easy way to make this nonzero is to choose $v(r) = u(r)\varphi(r)\varphi'(r)$, if φ is not constant. Then the curvature will be strictly positive.

If, on the other hand, φ is constant, then we can compute that $\nabla_X X = 0$ and also that the curvature of M vanishes. In this case, X is a pressure-constant flow in the terminology of Misiołek [7], who proved that such flows have nonpositive curvature operator $\tilde{\mathbf{R}}_X$ using a different curvature formula. □

4. Explicit formulas for curvature

Theorem 3.3 reduces the question of nonpositive curvature to certain special cases. To obtain further results, we compute an explicit formula for the curvature operator $\tilde{\mathbf{R}}_X$, where $X = u(r)\partial_\theta$ and the metric is $\mathrm{d}s^2 = \mathrm{d}r^2 + \varphi^2(r)\,\mathrm{d}\theta^2$. We assume the manifold is defined by the inequalities $0 \le r \le R$ and is either an annulus, a disc, or a sphere.

First we compute the curvature operator directly from the definition. We will use the convenient fact that on all three surfaces, every divergence-free vector field Y which is tangent to the boundary can be written as $Y = \operatorname{sgrad} g$, where g is constant on each boundary component. Because of the rotational symmetry, it is natural to expand g as a Fourier series $g(r,\theta) = \sum_{n=-\infty}^{\infty} g_n(r)\,\mathrm{e}^{\mathrm{i}n\theta}$, where $g_{-n}(r) = \overline{g}_n(r)$. We consider a particular component of this expansion, and let

$$Y_n = \operatorname{sgrad}(g_n(r)\,\mathrm{e}^{\mathrm{i}n\theta}). \tag{4.1}$$

If $n \neq 0$, then since $g_n(r)\,\mathrm{e}^{\mathrm{i}n\theta}$ must be constant on the boundary, we must have $g_n(r) = 0$ on the boundary. On the other hand, g_0 may be an arbitrary constant on each boundary component.

Proposition 4.1. *If $X = u(r)\partial_\theta$ on an annulus, disc, or sphere with a rotationally symmetric metric, then the curvature operator of the diffeomorphism group in direction X is given by $\tilde{\mathbf{R}}_X(Y_0) = 0$ and, if $n \neq 0$,*

$$\tilde{\mathbf{R}}_X(Y_n) = \mathbf{P}\Bigg(-\mathrm{i}nu'(r)v(r)g_n(r)\,\mathrm{e}^{\mathrm{i}n\theta}\partial_r - \frac{v(r)v'(r)}{\varphi(r)}g_n(r)\,\mathrm{e}^{\mathrm{i}n\theta}\partial_\theta - \mathrm{i}nu'(r)q_n(r)\,\mathrm{e}^{\mathrm{i}n\theta}\partial_r - \frac{\mathrm{i}nv(r)}{\varphi(r)}q_n(r)\,\mathrm{e}^{\mathrm{i}n\theta}\partial_r + \frac{v(r)}{\varphi(r)}q_n'(r)\,\mathrm{e}^{\mathrm{i}n\theta}\partial_\theta\Bigg), \tag{4.2}$$

where $v(r) \equiv u(r)\varphi'(r)$ and q_n is defined to be the solution of the Neumann problem

$$\frac{1}{\varphi(r)}\frac{\mathrm{d}}{\mathrm{d}r}\left(\varphi(r)\frac{\mathrm{d}q_n}{\mathrm{d}r}\right) - \frac{n^2}{\varphi^2(r)}q_n(r) = \frac{1}{\varphi(r)}\frac{\mathrm{d}}{\mathrm{d}r}(\varphi(r)v'(r)g_n(r)) + \frac{n^2u'(r)}{\varphi(r)}g_n(r) \tag{4.3}$$

with boundary condition $q_n' = 0$.

Proof. Since $\mathbf{P}(\nabla_X X) = 0$, formula (2.5) simplifies to

$$\tilde{\mathbf{R}}_X(Y) = \mathbf{P}(-\nabla_X \mathbf{P}(\nabla_Y X) + \nabla_{[X,Y]} X). \tag{4.4}$$

First, in case $n = 0$, we have $Y_0 = \operatorname{sgrad} g_0(r) = (g_0'(r)/\varphi(r))\partial_\theta$. We compute $\nabla_Y X = -g_0(r)u(r)\varphi'(r)\partial_r$, and since this is a gradient, we know $\mathbf{P}(\nabla_Y X) = 0$. Also, we obviously have $[X, Y] = 0$. Thus both terms in formula (4.4) vanish, so $\tilde{\mathbf{R}}_X(Y_0) = 0$.

In the remainder of the proof we suppose that $n \neq 0$, so that g_n vanishes on the boundary (if any). Then we can write, by formula (4.1),

$$Y_n(r, \theta) = -\frac{\mathrm{i}n}{\varphi(r)} g_n(r)\,\mathrm{e}^{\mathrm{i}n\theta}\partial_r + \frac{1}{\varphi(r)} g_n'(r)\,\mathrm{e}^{\mathrm{i}n\theta}\partial_\theta.$$

For brevity, we will use the abbreviation $e_n \equiv \mathrm{e}^{\mathrm{i}n\theta}$.

We first compute $\nabla_{Y_n} X$.

$$\nabla_{Y_n} X = v' g_n e_n \partial_r - \frac{\mathrm{i}nu'}{\varphi} g_n e_n \partial_\theta - \nabla(v g_n e_n). \tag{4.5}$$

Using formula (2.4) and the fact that $\mathbf{P}$ vanishes on gradients, we find that

$$\mathbf{P}(\nabla_{Y_n} X) = v' g_n e_n \partial_r - \frac{\mathrm{i}nu'}{\varphi} g_n e_n \partial_\theta - q_n' e_n\, \partial_r - \frac{\mathrm{i}n}{\varphi^2} q_n e_n \partial_\theta, \tag{4.6}$$

where $q_n(r)\,\mathrm{e}^{\mathrm{i}n\theta}$ is defined to be the solution of the Neumann problem

$$\Delta(q_n(r)\,\mathrm{e}^{\mathrm{i}n\theta}) = \operatorname{div}\left(v'(r) g_n(r)\,\mathrm{e}^{\mathrm{i}n\theta}\partial_r - \frac{\mathrm{i}nu'(r)}{\varphi(r)} g_n(r)\,\mathrm{e}^{\mathrm{i}n\theta}\partial_\theta\right).$$

Computing the Laplacian and divergence explicitly, we obtain formula (4.3). The condition that q_n' vanish on the boundary is a consequence of the fact that g_n vanishes on the boundary.

Having obtained formula (4.6), the other terms of formula (4.4) are straightforward to compute.

We obtain:

$$\begin{aligned}
\nabla_X \mathbf{P}(\nabla_{Y_n} X) &= in(uv' + vu')g_n e_n \partial_r + \frac{n^2 uu' + vv'}{\varphi} g_n e_n \partial_\theta \\
&\quad + in\left(u' + \frac{v}{\varphi}\right) q_n e_n \partial_r - \frac{v}{\varphi} q_n' e_n \partial_\theta - \nabla(\mathrm{i}nuq_n e_n),\\
\nabla_{[X,Y_n]} X &= \mathrm{i}nuv' g_n e_n \partial_r + \frac{n^2 uu'}{\varphi} g_n e_n \partial_\theta - \nabla(\mathrm{i}nuv g_n e_n).
\end{aligned}$$

Combining the two expressions, and using the fact that $\mathbf{P}$ vanishes on gradients, we obtain formula (4.2). □

We immediately obtain the following useful consequence of formula (4.2).

Proposition 4.2. *The curvature operator $\tilde{\mathbf{R}}_X$ is nonpositive if and only if, for every $n > 0$ and every vector field Y_n of the form* (4.1), *the sectional curvature $\mathbf{K}(X, Y_n)$ is nonpositive.*

Proof. Formula (4.2) shows that $\langle\langle \tilde{\mathbf{R}}(Y_n, X)X, Y_m\rangle\rangle = 0$ unless $m = -n$, by the usual orthogonality of Fourier series. Thus we can write

$$\langle\langle \tilde{\mathbf{R}}_X(Y), Y\rangle\rangle = \sum_{n=-\infty}^{\infty} \langle\langle \tilde{\mathbf{R}}(Y_n, X)X, \overline{Y}_n\rangle\rangle = 2\sum_{n=1}^{\infty} \mathbf{K}(X, Y_n).$$

The proposition is then obvious. □

In the next proposition we show how the seemingly complicated formula (4.2) leads to a much simpler expression for the sectional curvature.

Proposition 4.3. *If $n \neq 0$, the curvature $\mathbf{K}_n \equiv \langle\langle \tilde{\mathbf{R}}(Y_n, X)X, \overline{Y}_n\rangle\rangle$ is given by the formula*

$$\mathbf{K}_n = 2\pi \int_0^R (\varphi v'^2 |g_n|^2 + \overline{g}_n [n^2 u q_n - \varphi v' q_n'])\,\mathrm{d}r, \tag{4.7}$$

where v and q_n are as defined in Proposition 4.1.

Proof. Formula (4.2) implies

$$\begin{aligned}\mathbf{K}_n &= \int_M \frac{n^2}{\varphi^2} \overline{g}_n [\varphi u' v g_n + \varphi u' q_n + v q_n]\mu + \int_M \overline{g}_n' [v q_n' - v v' g_n]\mu \\ &= 2\pi \int_0^R n^2 \overline{g}_n \left[u' v g_n + u' q_n + \frac{1}{\varphi} v q_n\right] \mathrm{d}r + 2\pi \int_0^R \overline{g}_n' [\varphi v q_n' - \varphi v v' g_n]\,\mathrm{d}r.\end{aligned} \tag{4.8}$$

We can integrate one of these terms by parts to obtain

$$\begin{aligned}\int_0^R \overline{g}_n' \varphi v q_n'\,\mathrm{d}r &= \overline{g}_n \varphi v q_n' \Big|_0^R - \int_0^R \overline{g}_n v' \varphi q_n'\,\mathrm{d}r - \int_0^R \overline{g}_n v \frac{\mathrm{d}}{\mathrm{d}r}(\varphi q_n')\,\mathrm{d}r \\ &= -\int_0^R \overline{g}_n \left(v' \varphi q_n' + \frac{n^2}{\varphi} q_n + \frac{\mathrm{d}}{\mathrm{d}r}(\varphi v' g_n) + n^2 u' v g_n \right) \mathrm{d}r,\end{aligned}$$

using Eq. (4.3) and the fact that at both $r = 0$ and $r = R$, either g_n or φ vanishes.

Plugging this expression into (4.8), we get

$$\mathbf{K}_n = 2\pi \int_0^R \overline{g}_n [n^2 u' q_n - \varphi v' q_n']\,\mathrm{d}r - 2\pi \int_0^R v \frac{\mathrm{d}}{\mathrm{d}r}(\varphi v' \overline{g}_n g_n)\,\mathrm{d}r.$$

Another integration by parts establishes formula (4.7). □

The last part is to compute the function q_n more explicitly. The fact that we can do this is what enables us to get complete results. Our final formula for the curvature depends only on first integrals, which can be computed explicitly for any given φ, u, and g_n. There are three somewhat different solutions for the annulus, disc, and sphere, due to the different boundary conditions for q_n. First we need to define some auxiliary functions.

Definition 4.4. Let $\xi(r)$ be a function such that $\xi'(r) = 1/\varphi(r)$. There are three cases:

$$\xi(r) = \begin{cases} \int_0^r \frac{1}{\varphi(\rho)}\,\mathrm{d}\rho & \text{on the annulus,} \\ \ln r + \int_0^r \left(\frac{1}{\varphi(\rho)} - \frac{1}{\rho}\right)\mathrm{d}\rho & \text{on the disc,} \\ \ln r - \ln(R-r) + \int_0^r \left(\frac{1}{\varphi(\rho)} - \frac{1}{\rho} - \frac{1}{R-\rho}\right)\mathrm{d}\rho & \text{on the sphere.} \end{cases} \tag{4.9}$$

Given a function $g_n(r)$ vanishing on the boundary components (if any) of M, define for integers $n > 0$ functions H_n and J_n by the formulas

$$H_n(r) = \int_0^r [nu'(s) - v'(s)] g_n(s)\,\mathrm{e}^{n\xi(s)}\,\mathrm{d}s, \tag{4.10}$$

$$J_n(r) = \int_r^R [nu'(s) + v'(s)] g_n(s)\,\mathrm{e}^{-n\xi(s)}\,\mathrm{d}s. \tag{4.11}$$

Because $\varphi(0) = 0$ on the disc and sphere, we must have $\varphi'(0) = 1$ for smoothness; similarly, we must have $\varphi'(R) = -1$ on the sphere. So the expansions as written are to ensure that the integrands are smooth at 0 and R; the only singular behavior of ξ appears in the logarithmic terms. The conditions on ξ ensure that both of the integrals (4.10) and (4.11) are proper and well-defined regardless of which surface we are on.

Proposition 4.5. *The solution q_n of the problem* (4.3) *is given by*

$$q_n(r) = -\tfrac{1}{2}(\mathrm{e}^{-n\xi(r)} H_n(r) + \mathrm{e}^{n\xi(r)} J_n(r) + A_n\,\mathrm{e}^{n\xi(r)} + B_n\,\mathrm{e}^{-n\xi(r)}), \tag{4.12}$$

where A_n and B_n are constants given by

$$\begin{aligned} A_n &= \frac{H_n(R) + J_n(0)}{\mathrm{e}^{2n\xi(R)} - 1}, & B_n &= \frac{H_n(R) + \mathrm{e}^{2n\xi(R)} J_n(0)}{\mathrm{e}^{2n\xi(R)} - 1} & &(\textit{annulus}), \\ A_n &= \mathrm{e}^{-2n\xi(R)} H_n(R), & B_n &= 0 & &(\textit{disc}), \\ A_n &= 0, & B_n &= 0 & &(\textit{sphere}). \end{aligned}$$

Here $\xi(r)$, $H_n(r)$, and $J_n(r)$ are defined by Definition 4.4.

Proof. The main point is that solutions of the homogeneous equation

$$\varphi(r)\frac{\mathrm{d}}{\mathrm{d}r}\left(\varphi(r)\frac{\mathrm{d}}{\mathrm{d}r} q_n(r)\right) - n^2 q_n(r) = 0$$

are given by $q_n(r) = \mathrm{e}^{\pm n\xi(r)}$. Thus we can use the standard Green function approach to find the formula. Integration by parts yields the particular combinations of H_n and J_n. Finally the boundary conditions yield A_n and B_n: the difference in the formulas on the three surfaces is due to the fact that on the annulus, we have two boundary conditions $q_n'(0) = q_n'(R) = 0$, while on the sphere we have only the requirement that q_n be finite at $r = 0$ and $r = R$, and on the disc we have a combination of the two types. □

From these explicit formulas, we can obtain a very explicit formula for the curvature of the area-preserving diffeomorphism group.

Proposition 4.6. *The curvature* $\mathbf{K}_n = \langle\langle \tilde{\mathbf{R}}(Y_n, X)X, \overline{Y_n}\rangle\rangle$ *is given by*

$$\mathbf{K}_n = -n\pi \int_0^R (\overline{H_n'(r)}J_n(r) - \overline{J_n'(r)}H_n(r))\,\mathrm{d}r \\ -n\pi \begin{cases} \left(\dfrac{1}{\mathrm{e}^{2n\xi(R)}-1}\right)\left|H_n(R)+\overline{J_n(0)}\right|^2 + |J_n(0)|^2 & (\textit{annulus}),\\ \mathrm{e}^{-2n\xi(R)}|H_n(R)|^2 & (\textit{disc}),\\ 0 & (\textit{sphere}). \end{cases} \tag{4.13}$$

Proof. The formulas for curvature follow readily from formula (4.12) for q_n and formula (4.7) for the curvature. □

It is convenient to have the following simplification of the integral which appears in the curvature computation for all three surfaces.

Lemma 4.7. *On the annulus, disc, and sphere, we have*

$$\int_0^R (\bar{H}_n' J_n - \bar{J}_n' H_n)\,\mathrm{d}r = 2\,\mathrm{Re}\int_0^R \bar{H}_n' J_n\,\mathrm{d}r = -2\,\mathrm{Re}\int_0^R \bar{J}_n' H_n\,\mathrm{d}r. \tag{4.14}$$

Proof. This is just an integration by parts of one or the other term. On the annulus both equations are obvious because $\overline{H_n(0)} = J_n(R) = 0$. On the disc or the sphere, however, we have to check that $\lim_{r\to 0^+}\overline{H_n(r)}J_n(r) = 0$, and on the sphere, we have to check that $\lim_{r\to R^-}\overline{H_n(r)}J_n(r) = 0$. Both limit computations are essentially the same and involve an application of L'Hôpital's rule, which we can omit. □

5. The nonpositivity criterion

The general formula (4.13) gives the simplest known formula for the curvature of the area-preserving diffeomorphism group, in the special case of rotational flow. Although we do not expect to be able to write the integral (4.14) any more explicitly, we can still study its sign and determine necessary and sufficient conditions for it to be nonpositive.

The next lemma gives one part of the nonpositive curvature criterion.

Lemma 5.1. *If u and φ are real analytic functions on* $[0, r]$ *with* $X = u(r)\partial_\theta$ *and* $\mathrm{d}s^2 = \mathrm{d}r^2 + \varphi^2(r)\,\mathrm{d}\theta^2$ *the metric on a smooth manifold M, and if the curvature operator* $\tilde{\mathbf{R}}_X$ *is nonpositive, then the function* $Q = (u\varphi')'/u'$ *is defined for all* $r \in [0, R]$.

Proof. We will show that if Q is undefined for any $r_0 \in (0, R)$, then we have positive curvature. If $Q(r_0)$ is undefined, then the analytic function u' has a zero of order at least one more than v'. So we have

$$u(r) = u(r_0) + \frac{u^{(k+j)}(r_0)}{(k+j)!}(r-r_0)^{k+j} + \mathrm{O}((r-r_0)^{k+j+1}),$$

$$v(r) = v(r_0) + \frac{v^{(k)}(r_0)}{k!}(r - r_0)^k + \mathrm{O}((r - r_0)^{k+1})$$

for some integers $k \geq 1$, $j \geq 1$, with $v^{(k)}(r_0) \neq 0$ and $u^{(k+j)}(r_0) \neq 0$. It is possible that $r_0 = 0$ or $r_0 = R$; however, we can assume without loss of generality that $r_0 < R$, since any boundary issues at $r_0 = R$ are identical to those at $r_0 = 0$.

Choose an $\varepsilon > 0$ such that $r_0 + \varepsilon < R$, and define

$$g_1(r) = \begin{cases} 1 \text{ if } r_0 \leq r \leq r_0 + \varepsilon, \\ 0 \text{ otherwise.} \end{cases}$$

The fact that g_1 is not continuous does not affect the result since g_1 only appears in integrals; thus we can approximate it in the L^∞ norm by a smooth function.

First we suppose that ξ is nonsingular at r_0. Then the integral H_1 given by Eq. (4.10) can be approximated by

$$H_1(r) = -\frac{\mathrm{e}^{\xi(r_0)} v^{(k)}(r_0)}{k!}(r - r_0)^k + \mathrm{O}(\varepsilon^{k+1})$$

for $r \in [r_0, r_0 + \varepsilon]$, and since

$$J_1'(r) = -\frac{v^{(k)}(r_0)}{(k-1)!}\, \mathrm{e}^{-\xi(r_0)}(r - r_0)^{k-1} + \mathrm{O}(\varepsilon^k)$$

on this interval, we can compute using Lemma 4.7 the integral appearing in all three curvature formulas:

$$\int_0^R (\overline{H_1'(r)} J_1(r) - \overline{J_1'(r)} H_1(r))\, \mathrm{d}r = -\left(\frac{v^{(k)}(r_0)\varepsilon^k}{k!}\right)^2 + \mathrm{O}(\varepsilon^{2k+1}).$$

Using the fact that, to order $\mathrm{O}(\varepsilon^{k+1})$, we have

$$H_1(R) = -\frac{\mathrm{e}^{\xi(r_0)} v^{(k)}(r_0)\varepsilon^k}{k!} \quad \text{and} \quad J_1(0) = \frac{\mathrm{e}^{-\xi(r_0)} v^{(k)}(r_0)\varepsilon^k}{k!},$$

the curvature formula (4.13) can be seen to give positive numbers on each surface for sufficiently small $\varepsilon > 0$. We omit the details.

Now we consider the case where ξ is singular at r_0. So we must be on a disc or a sphere with $r_0 = 0$, and thus $\mathrm{e}^{\xi(r)} \approx r$. For $r \in [0, \varepsilon]$, we have

$$H_1(r) = -\frac{v^{(k)}(0)}{(k-1)!(k+1)} r^{k+1} + \mathrm{O}(\varepsilon^{k+2}),$$

and

$$J_1'(r) = -\frac{v^{(k)}(0)}{(k-1)!} r^{k-2} + \mathrm{O}(\varepsilon^{k-1}).$$

Using Lemma 4.7, we can compute

$$\int_0^R (\overline{H_1'(r)} J_1(r) - \overline{J_1'(r)} H_1(r))\,\mathrm{d}r = -\frac{(v^{(k)}(r_0)\varepsilon^k)^2}{(k-1)!(k+1)!} + \mathrm{O}(\varepsilon^{2k+1}).$$

So in this case as well, formula (4.13) for the disc or sphere yields positive curvature. □

The next proposition gives us useful information about the functions which satisfy the differential inequality $\varphi Q' + Q^2 \leq 1$, which will be used in the proof of the nonpositive curvature criterion.

Proposition 5.2. *Suppose Q is a real analytic function on* $[0, R]$ *such that*

$$\varphi(r)Q'(r) + Q^2(r) \leq 1 \tag{5.1}$$

for all $r \in (0, R)$. *Then for any* $0 < a < b < R$, *if* $|Q(r)| < 1$ *on* $[a, b]$, *we have the inequality*

$$\operatorname{arctanh} Q(b) - \operatorname{arctanh} Q(a) \leq \xi(b) - \xi(a). \tag{5.2}$$

In addition, we have the following:

- *If* $|Q(r)| > 1$, *then Q is strictly decreasing at r.*
- *If* $Q'(0) = 0$, *then either* $Q(r) < 1$ *for all* $r \in (0, R)$, *or* $Q(r) \equiv 1$ *on* $[0, R]$.
- *If* $Q'(R) = 0$, *then either* $Q(r) > -1$ *for all* $r \in (0, R)$, *or* $Q(r) \equiv -1$ *on* $[0, R]$.

Proof. The inequality (5.2) follows by integrating (5.1). The fact that $|Q(r)| > 1$ implies $Q'(r) < 0$ is obvious from (5.1).

If $Q'(0) = 0$, then (5.1) implies by continuity that $Q^2(0) \leq 1$. If $Q(0) < 1$ then (5.2) implies that $Q(r) < 1$ for all $r < R$. If $Q(0) = 1$ and $Q(r) \not\equiv 1$ for all $[0, R]$, then by analyticity, either $Q(r) < 1$ or $Q(r) > 1$ for all sufficiently small r. But it is not possible that $Q(r) > 1$ for all sufficiently small r, since then Q must be strictly decreasing. So $Q(r) < 1$ for some small r, and thus (5.2) implies that in fact $Q(r) < 1$ for all $r \in (0, R)$.

The argument for $Q'(R) = 0$ is identical. □

These computations culminate in the following criterion for a steady flow in two dimensions to have nonpositive curvature operator.

Theorem 5.3. *Suppose* $X = u(r)\partial_\theta$ *on an annulus, disc, or sphere M with metric* $\mathrm{d}s^2 = \mathrm{d}r^2 + \varphi^2(r)\,\mathrm{d}\theta^2$, *with u and φ real-analytic functions on* $[0, R]$ *with appropriate conditions at* 0 *and R to ensure smoothness on M, and with u nowhere zero.*

Define $v = \varphi' u$. *Then* $\tilde{\mathbf{R}}_X$ *is nonpositive if and only if the function* $Q \equiv v'/u'$ *is defined for all* $r \in [0, R]$ *and satisfies*

$$\varphi(r)Q'(r) + Q^2(r) \leq 1 \quad \textit{for all } r \in (0, R). \tag{5.3}$$

The basic idea of this proof is that the ratio J_n'/H_n' does not depend on g_n:

$$\frac{J_n'(r)}{H_n'(r)} = -\frac{n+Q(r)}{n-Q(r)}\,\mathrm{e}^{-2n\xi(r)}.$$

Therefore the integral (4.14) can be computed using integration by parts. In the simplest case where $H_n(R)=0$ and $J_n(0)=0$, all the boundary terms in the formula (4.13) vanish. Thus if $(n-Q)$ does not vanish anywhere,

$$\begin{aligned}\mathbf{K}_n &= 2n\pi\,\mathrm{Re}\int_0^R \overline{J_n'(r)}H_n(r)\,\mathrm{d}r\\ &= n\pi\int_0^R \frac{J_n'(r)}{H_n'(r)}\frac{\mathrm{d}}{\mathrm{d}r}(|H_n(r)|^2)\,\mathrm{d}r\\ &= n\pi\int_0^R \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{n+Q(r)}{n-Q(r)}\,\mathrm{e}^{-2n\xi(r)}\right)|H_n(r)|^2\,\mathrm{d}r\\ &= 2n^2\pi\int_0^R \frac{\mathrm{e}^{-2n\xi(r)}[\varphi(r)Q'(r)+Q^2(r)-n^2]}{\varphi(r)[n-Q(r)]^2}|H_n(r)|^2\,\mathrm{d}r. \qquad (5.4)\end{aligned}$$

The key idea then is that as long as the inequality $\varphi Q' + Q^2 \le n^2$ is satisfied for every $n>0$, the integral is nonpositive. If the boundary terms do not vanish, we can verify that they are also nonpositive. The main complication is the possibility that $Q(r_0)=n$ for some $r_0\in(0,R)$, and we deal with this case-by-case.

Proof. By Lemma 5.1, we know that Q must be defined everywhere for $\tilde{\mathbf{R}}_X$ to be nonpositive.

If for some r_0, $\varphi(r_0)Q'(r_0)+Q^2(r_0)>1$, then by continuity we have $\varphi Q'+Q^2>1$ on some interval (a,b). Choose a subinterval (c,d) such that $Q\neq 1$ on (c,d). Choose a function g_1 such that

$$\int_c^d (u'(r)-v'(r))\,\mathrm{e}^{\xi(r)}g_1(r)\,\mathrm{d}r = \int_c^d (u'(r)+v'(r))\,\mathrm{e}^{-\xi(r)}g_1(r)\,\mathrm{d}r = 0,$$

and so that

$$g_1(r)=0 \quad \text{for } r\notin(c,d), \quad \text{and} \quad g_1\not\equiv 0 \text{ on } (c,d).$$

Of course, there is an infinite-dimensional space of such functions.

Then we will have $H_1(R)=J_1(0)=0$, so on all three surfaces, the curvature formula is the same:

$$\mathbf{K}_1 = 2\pi\int_c^d |H_1(r)|^2\frac{\varphi(r)Q'(r)+Q^2(r)-1}{\varphi(r)[1-Q(r)]^2}\,\mathrm{d}r>0.$$

Thus by Proposition 4.2, the curvature operator $\tilde{\mathbf{R}}_X$ is not nonpositive.

Now we suppose that $\varphi Q' + Q^2 \leq 1$ for all $r \in (0, R)$; we will show that $\mathbf{K}_n \leq 0$ for every Y_n. The computation is slightly different on the three possible surfaces.

On the disc or sphere, we must have $Q'(0) = 0$, and thus Proposition 5.2 implies that either $Q(r) \equiv 1$ on $[0, R]$, or $Q(r) < 1$ for all $r \in (0, R)$. In case $Q \equiv 1$ and $n = 1$, we know by formula (4.10) that $H_1 \equiv 0$, so that $\mathbf{K}_1 = 0$ by formula (4.13). If $Q < 1$ on $(0, R)$, or if $n > 1$, then $Q(r) \neq n$ on $(0, R)$ and formulas (5.4) and (4.13) can be used to give

$$\mathbf{K}_n = 2n^2\pi \int_0^R \frac{e^{-2n\xi(r)}[\varphi(r)Q'(r) + Q^2(r) - n^2]}{\varphi(r)[n - Q(r)]^2} |H_n(r)|^2 \, dr - n\pi \begin{cases} \left(\frac{2n}{n-Q(R)}\right) e^{-2n\xi(R)} |H_n(R)|^2 & \text{(disc)}, \\ 0 & \text{(sphere)}. \end{cases}$$

The integral is clearly nonpositive. On the disc, $Q(R) < 1$; thus the boundary term is also nonpositive.

If $n = 1$, one is naturally concerned about the possibility that $Q(0) = 1$ (for the disc or sphere) or that $Q(R) = 1$ (for the sphere). However, the approximations $e^{\xi(r)} \approx r$ and $H_1(r) \approx r^3$ ensure that the integral is proper at $r = 0$, even if $Q(0) = 1$. Similar formulas ensure that the integral on the sphere is proper at $r = R$.

The annulus is more complicated, because there is no restriction on $Q(0)$ or $Q(R)$; thus the boundary terms are harder to control. We consider three cases: either $Q(0) < n$ and $Q(R) < n$; or $Q(R) > n$ and $Q(0) > n$; or $Q(0) \geq n$ and $Q(R) \leq n$. The case $Q(0) < n$ and $Q(R) > n$ is impossible by Proposition 5.2.

If $Q(0) < n$ and $Q(R) < n$, the computation (5.4) yields

$$\mathbf{K}_n = 2n^2\pi \int_0^R \frac{e^{-2n\xi(r)}[\varphi(r)Q'(r) + Q^2(r) - n^2]}{\varphi(r)[n - Q(r)]^2} |H_n(r)|^2 \, dr - n\pi \, e^{-2n\xi(R)} \left(\frac{2n}{n - Q(R)} |H_n(R)|^2 + \frac{1}{e^{2n\xi(R)} - 1} \left| H_n(R) + e^{2n\xi(R)} \overline{J_n(0)} \right|^2 \right),$$

which is nonpositive.

If $Q(0) > n$ and $Q(R) > n$, we do the same calculation as (5.4), except for reversing the roles of H_n and J_n:

$$\mathbf{K}_n = 2n^2\pi \int_0^R \frac{e^{2n\xi(r)}[\varphi(r)Q'(r) + Q^2(r) - n^2]}{\varphi(r)[n + Q(r)]^2} |J_n(r)|^2 \, dr - n\pi \left(\frac{2n}{n + Q(0)} |J_n(0)|^2 + \frac{1}{e^{2n\xi(R)} - 1} |H_n(R) + \overline{J_n(0)}|^2 \right),$$

which is again nonpositive.

In case $Q(0) \geq n$ and $Q(R) \leq n$, we have to do a little more. By Proposition 5.2, either $n = 1$ and $Q(r) \equiv 1$ on $[0, R]$, or $Q(r_0) = n$ for exactly one $r_0 \in [0, R]$. In the first case, we can as before conclude that $H_1 \equiv 0$ so that $\mathbf{K}_1 \leq 0$ by (4.13). In the second case, we do the same computation as in (5.4) using the function $[H_n(r) - H_n(r_0)]$ instead of $H_n(r)$, and obtain after some simplifications

$$\mathbf{K}_n = 2n^2\pi \int_0^R \frac{\mathrm{e}^{-2n\xi(r)}[\varphi(r)Q'(r) + Q^2(r) - n^2]}{\varphi(r)[n - Q(r)]^2} |H_n(r) - H_n(r_0)|^2 \,\mathrm{d}r$$
$$- n\pi \left(\frac{2n}{n - Q(R)} \mathrm{e}^{-2n\xi(R)} |H_n(R) - H_n(r_0)|^2 + \frac{2n}{Q(0) - n} |H_n(r_0)|^2 \right)$$
$$- n\pi \left(|H_n(r_0) + \overline{J_n(0)}|^2 + \frac{|H_n(R) + \overline{J_n(0)}|^2}{\mathrm{e}^{2n\xi(R)} - 1} - \mathrm{e}^{-2n\xi(R)} |H_n(R) - H_n(r_0)|^2 \right).$$

The terms on the first and second line are clearly nonpositive. The term on the last line is as well, using the triangle inequality

$$|H_n(R) - H_n(r_0)| \le |H_n(R) + \overline{J_n(0)}| + |H_n(r_0) + \overline{J_n(0)}|.$$

With the analysis of this last case, we are done showing that $\mathbf{K}_n$ is nonpositive for every g_n and every $n > 0$ if and only if the condition (5.3) for Q is satisfied. Thus by Proposition 4.2, we are done. □

6. Applications of the criterion

In this section we will derive some consequences of the criterion for nonpositive curvature proved in Theorem 5.3.

The simplest case is when Q is constant, which happens when M is a flat disc or cylinder.

Proposition 6.1. *If the metric on M is flat (i.e. $\varphi'' \equiv 0$) and $-1 \le \varphi' \le 1$, then every $X = u(r)\partial_\theta$ has nonpositive curvature operator $\tilde{\mathbf{R}}_X$.*

Proof. If φ' is a constant k, then $Q \equiv k$, regardless of u. So the criterion is that $k^2 \le 1$. □

For example, on a disc in $\mathbb{R}^2$, we have $\varphi' \equiv 1$. On a conical annulus embedded in $\mathbb{R}^3$, we have $\varphi' \equiv a$, with $0 < a < 1$. Thus both examples yield flows of nonpositive curvature. On an annulus with $\varphi' \equiv a$, with $a > 1$, no flow has nonpositive curvature operator.

The next proposition gives a local condition at the center of a disc for the curvature operator to be nonpositive. It also shows that if M is not flat, then in a small disc there will always be some rotational flows which satisfy the criterion of Theorem 5.3 and others which do not.

Proposition 6.2. *Suppose M is a disc or sphere such that $K(0)$, the curvature of M at the origin, is nonzero. Suppose $X = u(r)\partial_\theta$ satisfies $\tilde{\mathbf{R}}_X \le 0$. Then $u''(0) \ne 0$ and*

$$0 < \frac{u(0)K(0)}{u''(0)} \le 2. \tag{6.1}$$

Conversely, if $u''(0) \ne 0$ and the strict inequality

$$0 < \frac{u(0)K(0)}{u''(0)} < 2 \tag{6.2}$$

is satisfied, then for some $\varepsilon > 0$, X *satisfies* $\tilde{\mathbf{R}}_X \leq 0$ *on the disc*

$$\overline{B_\varepsilon(0)} = \{(r, \theta) | 0 \leq r \leq \varepsilon\}.$$

Proof. The proof is just based on Taylor expansions of u and φ at $r = 0$. We have

$$u(r) = u(0) + \tfrac{1}{2}u''(0)r^2 + \mathrm{O}(r^4), \qquad \varphi(r) = r - \tfrac{1}{6}K(0)r^3 + \mathrm{O}(r^5).$$

So $Q = (u\varphi')'/u'$ has the expansion

$$Q(r) = 1 - \frac{u(0)K(0)}{u''(0)} + \mathrm{O}(r^2),$$

if $u''(0) \neq 0$. On the other hand, if $u''(0) = 0$, then $Q(0)$ is undefined, which means $\tilde{\mathbf{R}}_X$ is not nonpositive. If $\tilde{\mathbf{R}}_X$ is nonpositive, then since $Q'(0) = 0$ and $\varphi(0) = 0$, the inequality (5.3) implies that $Q^2(0) \leq 1$. Thus $-1 \leq Q(0) \leq 1$, which implies the inequalities (6.1).

On the other hand, if (6.2) is satisfied, then $Q^2(0) < 1$. So for sufficiently small r,

$$\varphi(r)Q'(r) + Q^2(r) < 1$$

by continuity. Thus u satisfies the criterion of Theorem 5.3 for $0 \leq r \leq \varepsilon$, for some $\varepsilon \geq 0$. □

The following proposition gives one of the simplest examples.

Proposition 6.3. *If* $\varphi'(r) \neq -1$ *for all* $r \in [0, R]$, *then the velocity field determined by*

$$u(r) = \frac{1}{1 + \varphi'(r)}$$

has nonpositive curvature operator.

Proof. In this case, $u\varphi' = 1 - u$, so that $Q(r) \equiv -1$ and the criterion of Theorem 5.3 is automatically satisfied. □

Proposition 6.3 applies in particular on discs whose curvature is everywhere negative, since then $\varphi''(r) = -K(r)\varphi(r) > 0$ and thus $\varphi'(r) > 1$ for all r. It also applies on portions of spheres with everywhere positive curvature, since then $\varphi'(D) = -1$, where D is the diameter of the sphere, while $\varphi''(r) = -K(r)\varphi(r) < 0$ for all r implies that $\varphi'(r) > -1$ for all $r \leq R < D$.

However, there are no examples on an entire sphere of strictly positive curvature.

Proposition 6.4. *If* M *is a sphere with positive sectional curvature, then for every steady flow* X *with isolated nondegenerate zeroes, there is a divergence-free* Y *such that* $\langle\langle \tilde{\mathbf{R}}(Y, X)X, Y \rangle\rangle > 0$.

Proof. We can of course reduce this to the case where the metric is rotationally symmetric; then the requirement that the curvature of M is positive means that $\varphi''(r) < 0$ for all r. The definition of Q yields the equation

$$[Q(r) - \varphi'(r)]u'(r) = \varphi''(r)u(r). \tag{6.3}$$

Since X has no zeroes except at $r = 0$ and $r = R$, u is nowhere zero, and neither is $\varphi''(r)$ by the curvature condition. Thus $Q - \varphi'$ never vanishes.

Assume $\tilde{\mathbf{R}}_X \leq 0$. Then $\varphi(r)Q'(r) + Q^2(r) \leq 1$, so $Q(0) < 1$ as in Proposition 6.2. Similarly we find that $Q(R) > -1$.

Since $Q(0) - \varphi'(0) < 0$ and $Q(R) - \varphi'(R) > 0$, $Q - \varphi'$ must change sign in the interval $(0, R)$, a contradiction. □

The following proposition gives a qualitative picture of the types of flows that may have nonpositive curvature operator: if the manifold has positive or negative curvature, then u must be increasing or decreasing in r.

Proposition 6.5. *If the curvature of M is nowhere zero and if X is a steady vector field on M with $\tilde{\mathbf{R}}_X$ nonpositive, then $u'(r_0) \neq 0$ as long as $\varphi(r_0) \neq 0$.*

Proof. By Eq. (6.3), since Q must be defined at every point, $u'(r_0) = 0$ implies either $u(r_0) = 0$ or $\varphi''(r_0) = 0$. Since u is nowhere zero, we must have $\varphi''(r_0) = 0$, and since $\varphi''(r_0) = -K(r_0)\varphi(r_0)$ with $K(r_0) \neq 0$, we know $\varphi(r_0) = 0$. □

Of course, on a disc u must have a critical point at the origin, and on a sphere u must have critical points at the north and south poles, due to smoothness.

Now that we have some idea of what flows with nonpositive curvature operator look like, we can study the consequences this condition has for fluid stability. The physical interpretation of nonpositive curvature is a result of the Rauch comparison theorem, which allows us to estimate the growth of Jacobi fields along geodesics. Since geodesics in the area-preserving diffeomorphism group are flows of ideal fluids and Jacobi fields are linear Lagrangian perturbations, an upper bound on the curvature gives us a lower bound on the rate of growth of Jacobi fields.

The appropriate form of Rauch's theorem for volume-preserving diffeomorphism groups was proven by Misiołek [7].

Theorem 6.6 (Misiołek). *Let η be a geodesic in $\mathcal{D}_\mu(M)$ with tangent vector $\dot{\eta} = X(t) \circ \eta(t)$, and $Y(t)$ be a nonzero solution of the Jacobi equation*

$$\frac{\tilde{\mathbf{D}}^2}{\partial t^2}(Y(t) \circ \eta(t)) + \tilde{\mathbf{R}}(Y(t), X(t))X(t) \circ \eta = 0$$

with $Y(0) = 0$ and $(\tilde{\mathbf{D}}Y/\partial t)(0) = \dot{Y}_e$. If $\langle\langle \tilde{\mathbf{R}}(Y(t), X(t))X(t), Y(t)\rangle\rangle \leq 0$ for all t, then

$$\|Y(t)\|_{L^2} \geq \|\dot{Y}_e(0)\|_{L^2} \cdot t$$

for all $t \geq 0$.

Typically, many Lagrangian perturbations of an ideal fluid grow with time. (See examples in the author's previous work [9].) However, if the curvature operator $\tilde{\mathbf{R}}_X$ is nonpositive, then *every* Lagrangian perturbation grows with time, with growth at least $O(t)$. So we have uniform linear Lagrangian instability for flows $X = u(r)\partial_\theta$ if $\varphi Q' + Q^2 \leq 1$. It would be interesting to determine whether steady flows not satisfying this condition have any bounded Jacobi fields.

We can combine this result with previous work. The Arnold linear stability criterion says that if $X = u(r)\partial_\theta$ satisfies the condition

$$\frac{(\mathrm{d}/\mathrm{d}r)((1/\varphi)(\mathrm{d}/\mathrm{d}r)(\varphi^2 u))}{\varphi u} \neq 0 \quad \text{for any } r \in [0, R], \tag{6.4}$$

then every linearized Eulerian perturbation remains bounded in time. (Swaters [11] contains a general proof.) A theorem of the author [9] then says that any linearized Lagrangian perturbation of such a flow must grow at most quadratically in time. So if the conditions (5.3) and (6.4) are both satisfied, then every linearized Lagrangian perturbation $Y(t)$ of the fluid flow satisfies

$$at \leq \|Y(t)\|_{L^2} \leq bt^2 \quad \text{for } t \text{ sufficiently large},$$

for some constants a and b.

7. Related questions

By the theorem of Rouchon [10], we know that for any divergence-free vector field on an arbitrary manifold M, the curvature operator $\tilde{\mathbf{R}}_X$ takes on negative values if X is not a Killing field, and is nonnegative if X is a Killing field; it is never strictly positive. The result of this paper is almost complementary; if X is a steady solution of the Euler equation on a surface, then the curvature operator takes positive values if X does not satisfy the criteria of Theorem 5.3 and is nonpositive if it does; it is never strictly negative.

As noted above, it seems possible that the various restrictions imposed to obtain the Structure Theorem 3.3 and nonpositivity Theorem 5.3, such as the existence of a global stream function, the isolated and nondegenerate zeroes, and analyticity, may be unnecessary. On the other hand, it is conceivable that some new phenomena may arise if X is a harmonic vector field, or if X vanishes on a curve. These would be surprising and interesting.

Two very natural questions arise, however. First, what happens in three or more dimensions? We still expect to obtain $X\langle X, X\rangle = 0$, but we no longer expect to get such an explicit formula for X or the metric, and we probably also cannot get a formula for the inverse Laplacian in quadratures. The three-dimensional case therefore is probably quite different, and would be very interesting to study.

Second, what happens if X is not assumed to be a steady flow? It is conceivable that an arbitrary divergence-free vector field could have a curvature operator $\tilde{\mathbf{R}}_X$ that is strictly negative. A solution $X(t)$ of the nonsteady Euler equation for which the curvature operator was bounded above by some negative constant would have exponential growth of all Jacobi fields, by the Rauch comparison theorem. If this were possible, it would be the first time geometric methods rigorously predicted exponential instability.

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References

[1] T.A. Arakelyan, G.K. Savvidy, Geometry of a group of area-preserving diffeomorphisms, Phys. Lett. B 223 (1) (1989) 41–46.
[2] V.I. Arnold, Sur la geometrie differentielle des groupes de Lie de dimension infinie et ses applications a l'hydrodynamique des fluids parfaits, Ann. Inst. Fourier (Grenoble) 16 (1966) 319–361.
[3] V.I. Arnold, B.A. Khesin, Topological Methods in Hydrodynamics, Springer-Verlag, New York, 1998.
[4] J.S. Dowker, M.Z. Wei, Area-preserving diffeomorphisms and the stability of the atmosphere, Classical Quant. Gravity 7 (12) (1990) 2361–2365.
[5] A.M. Lukatsky, Structure of the curvature tensor of the group of measure-preserving diffeomorphisms of a compact two-dimensional manifold, Siberian Math. J. 29 (6) (1988) 947–951.
[6] A.M. Lukatsky, On the curvature of the diffeomorphisms group, Ann. Global Anal. Geom. 11 (2) (1993) 135–140.
[7] G.K. Misiołek, Stability of flows of ideal fluids and the geometry of the group of diffeomorphisms, Indiana Univ. Math. J. 42 (1) (1993) 215–235.
[8] S.C. Preston, Eulerian and Lagrangian stability of fluid motions, Ph.D. Thesis, SUNY Stony Brook, 2002.
[9] S.C. Preston, For ideal fluids, Eulerian and Lagrangian instabilities are equivalent, Geom. Funct. Anal. 14 (5) (2004).
[10] P. Rouchon, The Jacobi equation, Riemannian curvature and the motion of a perfect incompressible fluid, Eur. J. Mech. B Fluids 11 (3) (1992) 317–336.
[11] G. Swaters, Introduction to Hamiltonian Fluid Dynamics and Stability Theory, Chapman & Hall/CRC Press, Boca Raton, FL, 2000.
[12] K. Yoshida, Riemannian curvature on the group of area-preserving diffeomorphisms (motions of fluid) of 2-sphere, Physica D 100 (3/4) (1997) 377–389.